

ON THE ADJUNCTION FORMULA FOR 3-FOLDS IN CHARACTERISTIC $p > 5$

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ABSTRACT. In this article we prove a relative Kawamata-Viehweg vanishing-type theorem for PLT 3-folds in characteristic $p > 5$. We use this to prove the normality of minimal log canonical centers and the adjunction formula for codimension 2 subvarieties on \mathbb{Q} -factorial 3-folds in characteristic $p > 5$.

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1. INTRODUCTION

Let (X, Δ) be a log canonical pair and W a minimal log canonical center, then (under mild technical assumptions) by Kawamata's celebrated subadjunction theorem, it is known that W is normal and we can write $(K_X + \Delta)|_W = K_W + \Delta_W$ where (W, Δ_W) is Kawamata log terminal [Kaw98] (see also [Kaw97b] and the references therein). The proof of this result is based on the Kawamata-Viehweg vanishing theorem and Hodge theory. These results are known to fail in characteristic $p > 0$ and therefore one may expect that Kawamata's subadjunction also fails in this context. It should however be noted that related results have been obtained in the closely related context of F -singularities (see for example [Sch09], [HX15] and [Das15]) and that the minimal model program has been established for 3-folds in characteristic $p > 5$ (see [HX15] and [Bir13]). In particular [HX15] exploits the fact that PLT singularities in dimension 3 and characteristic $p > 5$ are closely related to the analogous notion of purely F -regular singularities. In this paper, using the results from [HX15] and [Bir13], we show that in dimension 3 and characteristic $p > 5$ a relative version of the Kawamata-Viehweg vanishing theorem

holds and we use this to establish that (under some mild technical conditions) the analog of Kawamata's subadjunction result holds.

Theorem (Theorem 3.5). *Let $f : (X, S+B \geq 0) \rightarrow Z$ be either a pl-divisorial contraction or a pl-flipping contraction (cf. Definition 3.1) such that $\dim X = 3$, $\text{char } p > 5$ and $S = \lfloor S+B \rfloor$ is reduced and irreducible. If the maximum dimension of the fibers of f is 1, then $R^i f_* \mathcal{O}_X(-S) = 0$ for all $i > 0$.*

This result allows us to prove the normality of the minimal LC centers for 3-folds.

Theorem (Theorem 3.6, 4.9). *Let (X, Δ) be a \mathbb{Q} -factorial 3-fold log canonical pair such that X has Kawamata Log Terminal singularities. If W is a minimal log canonical center of (X, Δ) , then W is normal. If moreover the coefficients of Δ belong to a DCC set $I \subseteq [0, 1]$ and $\text{char } k > \max\{5, \frac{2}{\delta}\}$, where $\delta > 0$ is the minimum of the set $D(I) \cap (0, 1]$ (where $D(I)$ is defined in 4.1), then the following hold:*

- (1) *There exists effective \mathbb{Q} -divisors Δ_W and M_W on W such that $(K_X + \Delta)|_W \sim_{\mathbb{Q}} K_W + \Delta_W + M_W$. Moreover, if $\Delta = \Delta' + \Delta''$ with Δ' (resp. Δ'') the sum of all irreducible components which contain (resp. do not contain) W , then M_W is determined only by the pair (X, Δ') .*
- (2) *There exists an effective \mathbb{Q} -divisor M'_W such that $M'_W \sim_{\mathbb{Q}} M_W$ and the pair $(W, \Delta_W + M'_W)$ is KLT.*

All of the results in this article hold in characteristic $p > 5$ unless stated otherwise. We will use the standard terminologies and notations from [KM98]. We also use the abbreviations: LC for log canonical, KLT for Kawamata log terminal, PLT for purely log terminal, DLT for divisorially log terminal, NLC for non-log canonical centers, NKLT centers for non-Kawamata log terminal centers, and lct for log canonical thresholds. If (X, Δ) is LC, then the NKLT centers are also known as log canonical centers or LC centers.

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2. PROPERTIES OF LOG CANONICAL CENTERS

In this section we establish some basic properties of LC centers.

Lemma 2.1. *Let X be a \mathbb{Q} -factorial KLT 3-fold and $(X, \Delta \geq 0)$ a log canonical pair. Let W_1 and W_2 be two log canonical centers of (X, Δ) . Then every irreducible component of $W_1 \cap W_2$ is a log canonical center of (X, Δ) .*

Proof. There are three cases depending on the codimension of W_1 and W_2 .

Case I: $\text{codim}_X W_1 = \text{codim}_X W_2 = 1$. In this case W_1 and W_2 are components of Δ . Let $\Delta = W_1 + W_2 + \bar{\Delta}$. Then by adjunction we have

$$(K_X + W_1 + W_2 + \bar{\Delta})|_{W_1^n} = K_{W_1^n} + \text{Diff}_{W_1^n}(\bar{\Delta}) + W_2|_{W_1^n},$$

where $W_1^n \rightarrow W_1$ is the normalization. By localizing at the generic point of an irreducible component of $W_1 \cap W_2$ we reduce to a surface problem. Now, on a surface in characteristic $p > 0$, the relative Kawamata-Viehweg vanishing and Kollár's connectedness theorem hold (see [Kol13, 10.13] and [Das15, 3.1]). Thus on a surface the intersection of two LC centers is a LC center and we are done by the usual argument (cf. [Kaw97a, Proposition 1.5]).

Case II: $\text{codim}_X W_1 = 1$ and $\text{codim}_X W_2 = 2$. Since X is \mathbb{Q} -factorial, $(X, (1 - \epsilon)\Delta)$ is KLT for any $0 < \epsilon < 1$. Thus by [Bir13, 7.7], there exists a \mathbb{Q} -factorial model $f' : X' \rightarrow X$ of relative Picard number $\rho(X'/X) = 1$ such that $\text{Ex}(f')$ is the unique exceptional divisor E' with center W_2 , and

$$(2.1) \quad K_{X'} + E' + W'_1 + \Delta' = f'^*(K_X + \Delta),$$

where $\Delta' \geq 0$, and W'_1 is the strict transform of W_1 under f' .

Since W'_1 and E' are \mathbb{Q} -Cartier, they intersect along a curve (possibly reducible). Let C' be an irreducible component of $W'_1 \cap E'$. Then by Case I, C' is a LC center of $(X', E' + W'_1 + \Delta' \geq 0)$. Since every irreducible component of $W_1 \cap W_2$ is dominated by an irreducible component of $W'_1 \cap E'$, we are done by relation (2.1).

Case III: $\text{codim}_X W_1 = \text{codim}_X W_2 = 2$. Again, since X is \mathbb{Q} -factorial, $(X, (1 - \epsilon)\Delta)$ is KLT for any $0 < \epsilon < 1$. Thus by [Bir13, 7.7], there exists a \mathbb{Q} -factorial model $f' : X' \rightarrow X$ extracting two exceptional divisors (one at a time) E'_1 and E'_2 such that $E'_1 \cap E'_2 \neq \emptyset$, $f'(E'_1) = W_1$ and $f'(E'_2) = W_2$, and

$$(2.2) \quad K_{X'} + E'_1 + E'_2 + \Delta' = f'^*(K_X + \Delta).$$

Since E'_1 and E'_2 are \mathbb{Q} -Cartier, they intersect along a curve (possibly reducible). Let C' be an irreducible component of $E'_1 \cap E'_2$. Then by Case I, C' is a LC center of $(X', E'_1 + E'_2 + \Delta' \geq 0)$. Since every irreducible component of $W_1 \cap W_2$ is dominated by an irreducible component of $E'_1 \cap E'_2$, we are done by relation (2.2). \square

The following proposition is a characteristic $p > 5$ version of Fujino's adjunction theorem for DLT pairs (see [Cor07, 3.9.2] and [Kol13, 4.16]) on a \mathbb{Q} -factorial 3-fold.

Proposition 2.2 (DLT Adjunction). *Let $(X, \Delta \geq 0)$ be a \mathbb{Q} -factorial DLT n -fold with $n \leq 3$ such that $\Delta = D_1 + D_2 + \cdots + D_r + B$ and $\lfloor \Delta \rfloor = D_1 + D_2 + \cdots + D_r$, where the D_i 's are prime divisors. Assume that $\text{char } p > 5$. Then the following hold:*

- (1) *The s -codimensional log canonical centers of (X, Δ) are exactly the irreducible components of the various intersections $D_{i_1} \cap \cdots \cap D_{i_s}$ for some $\{i_1, \dots, i_s\} \subseteq \{1, \dots, r\}$.*
- (2) *Every irreducible component of $D_{i_1} \cap \cdots \cap D_{i_s}$ is normal and of pure codimension s .*
- (3) *Let W be a log canonical center of (X, Δ) , then there exists an effective \mathbb{Q} -divisor $\Delta_W \geq 0$ on W such that $(K_X + \Delta)|_W \sim_{\mathbb{Q}} K_W + \Delta_W$ and (W, Δ_W) is DLT.*
- (4) *If $D_i \cap D_j = \emptyset$ for all $i \neq j$, then (X, Δ) is in fact PLT.*

Proof. The result is well known in dimension ≤ 2 . (1) follows from the proof in [Kol13, Theorem 4.16].

Since X is \mathbb{Q} -factorial, (X, D_i) is also PLT and then by adjunction $(D_i^n, \text{Diff}_{D_i^n})$ is KLT, where $D_i^n \rightarrow D_i$ is the normalization. Since $\text{Diff}_{D_i^n}$ has standard coefficients, by [Har98] and [HX15, 3.1], $(D_i^n, \text{Diff}_{D_i^n})$ is strongly F -regular in characteristic $p > 5$. Then by [HX15, 4.1] and [Das15, 4.1, 5.4], D_i is normal. This proves that every irreducible component of $\lfloor \Delta \rfloor$ is normal and hence (2) holds for $s = 1$.

It is easy to see that $(D_i, \text{Diff}_{D_i}(\Delta - D_i))$ is DLT, and so D_i is a \mathbb{Q} -factorial surface by [FT12, 6.3]. (2) and (3) now follow from the result in dimension 2. (4) is immediate. \square

3. VANISHING THEOREM AND MINIMAL LOG CANONICAL CENTERS

In this section we will prove a relative vanishing theorem and then use it to prove the normality of minimal log canonical centers.

Definition 3.1. Let $f : X \rightarrow Z$ be a projective birational morphism between normal quasi-projective varieties with relative Picard number $\rho(X/Z) = 1$. Let $(X, S + B \geq 0)$ be a \mathbb{Q} -factorial PLT pair such $\lfloor S + B \rfloor = S$ is irreducible, and $-S$ and $-(K_X + S + B)$ are both f -ample.

- (1) If $\dim \text{Ex}(f) = \dim X - 1$, then $f : X \rightarrow Z$ is called a *pl-divisorial contraction*.
- (2) If $\dim \text{Ex}(f) < \dim X - 1$, then $f : X \rightarrow Z$ is called a *pl-flipping contraction*.

Proposition 3.2. *Let $(X, S + B)$ be a \mathbb{Q} -factorial 3-fold PLT pair, where S is a prime Weil divisor. Assume that $(p^e - 1)(K_X + S + B)$ is an integral Weil divisor for some $e > 0$. Then there exists an integer $e_0 \gg 0$ such that the following sequence*

(3.1)

$$0 \longrightarrow \mathcal{B}_{ne_0} \longrightarrow F_*^{ne_0} \mathcal{O}_X((1 - p^{ne_0})(K_X + B) - p^{ne_0}S) \xrightarrow{\phi_{ne_0}} \mathcal{O}_X(-S) \longrightarrow 0$$

is exact at all codimension 2 points of X contained in S , for all $n \geq 1$, where ϕ_{ne_0} is defined by the trace map (see [Sch14] and [Pat14]) and \mathcal{B}_{ne_0} is the kernel of ϕ_{ne_0} .

Proof. By Proposition 2.2, S is normal. Since the question is local on X , we may assume that X is affine. Then by [HX15, 2.13], we can choose an effective \mathbb{Q} -Cartier divisor $G \geq 0$ not containing S and with sufficiently small coefficients such that $K_X + S + B + G$ is \mathbb{Q} -Cartier with index not divisible by p .

Localizing X at a codimension 2 point of X contained in S , we may assume that X is an excellent surface. Then by adjunction we have $(K_X + S + B + G)|_S = K_S + B_S + G|_S$, where B_S is the Different. Since $(X, S + B)$ is PLT, (S, B_S) is KLT by adjunction. Now, (S, B_S) is strongly F -regular by [HX15, 2.2], since S is a smooth curve. Since the coefficients of G are sufficiently small, $(S, B_S + G|_S)$ is also strongly F -regular. Therefore we get the following surjection

$$F_*^e \mathcal{O}_S((1 - p^e)(K_S + B_S + G|_S)) \twoheadrightarrow \mathcal{O}_S,$$

for all $e \gg 0$ and sufficiently divisible.

We have the following commutative diagram

$$(3.2) \quad \begin{array}{ccc} F_*^e \mathcal{O}_X((1 - p^e)(K_X + S + B + G)) & \twoheadrightarrow & F_*^e \mathcal{O}_S((1 - p^e)(K_S + B_S + G|_S)) \\ \downarrow & & \downarrow \\ \mathcal{O}_X & \twoheadrightarrow & \mathcal{O}_S \end{array}$$

To see the surjectivity of the top arrow note that since F_*^e is exact, it suffices to show that $(1 - p^e)(K_X + S + B + G)|_S = (1 - p^e)(K_S + B_S + G|_S)$, and since $(1 - p^e)(K_X + S + B + G)$ and $(1 - p^e)(K_S + B_S + G|_S)$ are Cartier for $e \gg 0$, it suffices to show that this equality holds at codimension 1 points of S , but this is clear since $(K_X + S + B + G)|_S = K_S + B_S + G|_S$. Since the ring \mathcal{O}_X is local, the surjectivity of the second vertical map (along with Nakayama's Lemma) implies the surjectivity of the first vertical map, i.e.,

$$(3.3) \quad F_*^{ne_0} \mathcal{O}_X((1 - p^{ne_0})(K_X + S + B + G)) \twoheadrightarrow \mathcal{O}_X$$

is surjective for all $n \geq 1$, where $e_0 \gg 0$ is sufficiently divisible.

Since the map (3.3) factors through $F_*^{ne_0} \mathcal{O}_X((1 - p^{ne_0})(K_X + B))$, we get the following surjectivity

$$(3.4) \quad F_*^{ne_0} \mathcal{O}_X((1 - p^{ne_0})(K_X + B)) \xrightarrow{\psi_{ne_0}} \mathcal{O}_X.$$

Let s be a pre-image of 1 under ψ_{ne_0} , then we get the following splitting of ψ_{ne_0}

$$(3.5) \quad \mathcal{O}_X \xrightarrow{\cdot s} F_*^{ne_0} \mathcal{O}_X((1 - p^{ne_0})(K_X + B)) \xrightarrow{\psi_{ne_0}} \mathcal{O}_X.$$

Twisting (3.5) by $\mathcal{O}_X(-S)$ and taking reflexive hulls we get the following splitting

$$(3.6) \quad \mathcal{O}_X(-S) \longrightarrow F_*^{ne_0} \mathcal{O}_X((1 - p^{ne_0})(K_X + B) - p^{ne_0}S) \longrightarrow \mathcal{O}_X(-S).$$

In particular the morphism

$$(3.7) \quad F_*^{ne_0} \mathcal{O}_X((1 - p^{ne_0})(K_X + B) - p^{ne_0}S) \longrightarrow \mathcal{O}_X(-S)$$

is surjective for all $n \geq 1$.

□

Remark 3.3. In Proposition 3.2, if we further assume that the coefficients of B are in the standard set $I = \{1 - \frac{1}{n} : n \geq 1\}$, then it follows that the sequence (3.1) is exact at all codimension 2 points of X . Indeed, by localizing at a codimension 2 point $P \in X \setminus S$, we may assume that (X, B) is an excellent surface. In this case the RHS of the sequence (3.1) takes the following form

$$(3.8) \quad F_*^{ne_0} \mathcal{O}_X((1 - p^{ne_0})(K_X + B)) \xrightarrow{\phi_{ne_0}} \mathcal{O}_X.$$

Since (X, B) is a PLT surface and $\lfloor B \rfloor = 0$, (X, B) is KLT. Thus by [Har98] and [HX15, 3.1], (X, B) is strongly F -regular in char $p > 5$. Since the coefficients of G are sufficiently small, $(X, B + G)$ is also strongly F -regular. Therefore we get the following surjectivity

$$(3.9) \quad F_*^{ne_0} \mathcal{O}_X((1 - p^{ne_0})(K_X + B + G)) \longrightarrow \mathcal{O}_X,$$

for some $e_0 \gg 0$ and sufficiently divisible, and for all $n \geq 1$.

Since the map in (3.9) factors through $F_*^{ne_0} \mathcal{O}_X((1 - p^e)(K_X + B))$, we get the following surjectivity

$$(3.10) \quad F_*^{ne_0} \mathcal{O}_X((1 - p^{ne_0})(K_X + B)) \xrightarrow{\phi_{ne_0}} \mathcal{O}_X.$$

Remark 3.4. In Proposition 3.2, we can further show that the sequence (3.1) is exact at all codimension 3 points of X contained in S , if the coefficients of B are in the standard set $I = \{1 - \frac{1}{n} : n \geq 1\}$. Indeed, by localizing at a codimension 3 point of X contained in S , we may assume that X is an excellent 3-fold. Then (S, B_S) is a KLT surface pair. By [Har98] and [HX15, 3.1], (S, B_S) is strongly F -regular in char $p > 5$. The rest of the proof runs without any changes.

Theorem 3.5. *Let $f : (X, S+B \geq 0) \rightarrow Z$ be either a pl-divisorial contraction or a pl-flipping contraction. If the maximum dimension of the fibers of f is 1, then $R^i f_* \mathcal{O}_X(-S) = 0$ for all $i > 0$.*

Proof. Since X is \mathbb{Q} -factorial, by perturbing the coefficients of B we may assume that $(p^e - 1)(K_X + S + B)$ is an integral Weil divisor for some $e > 0$. Since f is birational and $\text{Ex}(f) \subseteq \text{Supp}(S)$, it is enough to show that $R^i f_* \mathcal{O}_X(-S) = 0$ in a neighborhood of $f(S)$. Thus by restricting $(X, S+B)$ on a suitable neighborhood of S and by Proposition 3.2, we may assume that the following sequence is exact at all codimension 2 points of X

$$(3.11) \quad 0 \longrightarrow \mathcal{B}_e \longrightarrow F_*^e \mathcal{O}_X((1 - p^e)(K_X + B) - p^e S) \xrightarrow{\phi_e} \mathcal{O}_X(-S) \longrightarrow 0,$$

for all $e \gg 0$ and sufficiently divisible.

The sequence (3.11) can be split into the following two exact sequences

$$(3.12) \quad 0 \longrightarrow \mathcal{B}_e \longrightarrow F_*^e \mathcal{O}_X((1 - p^e)(K_X + B) - p^e S) \xrightarrow{\phi_e} \text{Im}(\phi_e) \longrightarrow 0$$

and

$$(3.13) \quad 0 \longrightarrow \text{Im}(\phi_e) \longrightarrow \mathcal{O}_X(-S) \longrightarrow \mathcal{Q}_e \longrightarrow 0,$$

where \mathcal{Q}_e is the corresponding quotient.

Pushing forward the exact sequence (3.12) by f_* we get

$$(3.14) \quad R^i f_*(F_*^e \mathcal{O}_X((1 - p^e)(K_X + B) - p^e S)) \rightarrow R^i f_* \text{Im}(\phi_e) \rightarrow R^{i+1} f_* \mathcal{B}_e.$$

Now $R^{i+1} f_* \mathcal{B}_e = 0$ for all $i > 0$, since the maximum dimension of the fiber of f is 1.

Let r be the index of $K_X + S + B$ and $H = -(K_X + S + B)$. By the division algorithm, there exist integers $k \geq 0$ and $0 \leq b < r$ such that $(p^e - 1) = r \cdot k + b$. Then by Serre vanishing

$$R^i f_*(F_*^e \mathcal{O}_X((1 - p^e)(K_X + B) - p^e S)) = F_*^e(R^i f_* \mathcal{O}_X(k \cdot r H - b(K_X + S + B) - S)) = 0,$$

for all $e \gg 0$ and sufficiently divisible, and $i > 0$, since H is f -ample.

Thus from (3.14) we get

$$(3.15) \quad R^i f_* \operatorname{Im}(\phi_e) = 0,$$

for all $i > 0$.

Again, pushing forward the exact sequence (3.13) by f_* we get

$$(3.16) \quad R^i f_* \operatorname{Im}(\phi_e) \rightarrow R^i f_* \mathcal{O}_X(-S) \rightarrow R^i f_* \mathcal{Q}_e.$$

$R^i f_* \mathcal{Q}_e = 0$ for all $i > 0$, since \mathcal{Q}_e is supported at finitely many points, by (3.11). Thus we have

$$(3.17) \quad R^i f_* \mathcal{O}_X(-S) = 0,$$

for all $i > 0$. □

Theorem 3.6. *Let (X, Δ) be a \mathbb{Q} -factorial 3-fold log canonical pair such that X has KLT singularities. If W is a minimal log canonical center of (X, Δ) , then W is normal.*

Proof. Since X is \mathbb{Q} -factorial and KLT, $(X, (1-\epsilon)\Delta)$ is KLT for any $0 < \epsilon < 1$, and all log canonical centers of (X, Δ) are contained in Δ . Then by Reid's Tie Breaking trick (see [Cor07, 8.7.1]) we may assume that W is the unique log canonical center of (X, Δ) with a unique divisor over X of discrepancy -1 . There are two cases depending on the codimension of W .

Case I: $\operatorname{codim}_X(W) = 1$. Since X is \mathbb{Q} -factorial, (X, W) is log canonical. By adjunction $(K_X + W)|_{W^n} = K_{W^n} + \operatorname{Diff}_{W^n}$, where $W^n \rightarrow W$ is the normalization and $(W^n, \operatorname{Diff}_{W^n})$ is KLT. Thus by [Har98] and [HX15, 3.1], $(W^n, \operatorname{Diff}_{W^n})$ is strongly F -regular in characteristic $p > 5$. Then $W^n = W$, i.e., W is normal by [HX15, 4.1] or [Das15, 4.1].

Case II: $\operatorname{codim}_X(W) = 2$. Let $f : (Y, S + \Delta') \rightarrow (X, \Delta)$ be an extraction of the unique exceptional divisor S over X such that

$$K_Y + S + \Delta' = f^*(K_X + \Delta).$$

Note that $-S$ is f -ample. Since $(Y, S + \Delta')$ is PLT, S is normal by Proposition 2.2. Also, since Y is \mathbb{Q} -factorial, (Y, S) is PLT.

Consider the following exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-S) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

Since W is contained in the support of Δ , $\Delta' \cap S \neq \emptyset$, and hence $-(K_Y + S)$ is f -ample. Thus $f : (Y, S) \rightarrow X$ is a pl-divisorial contraction. Then by Theorem 3.5, $R^1 f_* \mathcal{O}_Y(-S) = 0$, and we get the following exact sequence

$$(3.18) \quad 0 \longrightarrow f_* \mathcal{O}_Y(-S) \longrightarrow f_* \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_S \longrightarrow 0.$$

Since $f_* \mathcal{O}_Y(-S) = \mathcal{I}_W$ and $f_* \mathcal{O}_Y = \mathcal{O}_X$, we get

$$(3.19) \quad 0 \longrightarrow \mathcal{I}_W \longrightarrow \mathcal{O}_X \longrightarrow f_* \mathcal{O}_S \longrightarrow 0.$$

Now $\mathcal{O}_X \twoheadrightarrow f_* \mathcal{O}_S$ factors in the following way

$$(3.20) \quad \begin{array}{ccc} & \mathcal{O}_W & \\ \nearrow & \searrow & \\ \mathcal{O}_X & \longrightarrow & f_* \mathcal{O}_S \end{array}$$

$\nu_* \mathcal{O}_{W^n}$

where $\nu : W^n \rightarrow W$ is the normalization morphism.

Hence $\mathcal{O}_W = \nu_* \mathcal{O}_{W^n}$, i.e. W is normal. □

4. ADJUNCTION FORMULA

In this section we will prove an adjunction formula for 3-folds in characteristic $p > 5$. To start with we will need the following definitions and results.

Definition 4.1 (DCC sets). We say that a set I of real numbers satisfies the *descending chain condition* or DCC, if it does not contain any infinite strictly decreasing sequence. For example,

$$I = \left\{ \frac{r-1}{r} : r \in \mathbb{N} \right\}$$

satisfies the DCC.

Let $I \subseteq [0, 1]$. We define

$$I_+ := \{j \in [0, 1] : j = \sum_{p=1}^l i_p \text{ for some } i_1, i_2, \dots, i_l \in I\}$$

and

$$D(I) := \{a \leq 1 : a = \frac{m-1+f}{m}, m \in \mathbb{N}, f \in I_+\}.$$

Lemma 4.2. [MP04, 4.4] *Let $I \subseteq [0, 1]$. Then*

- (1) $D(D(I)) = D(I) \cup \{1\}$.
- (2) I satisfies DCC if and only if \bar{I} satisfies the DCC, where \bar{I} is the closure of I .
- (3) I satisfies DCC if and only if $D(I)$ satisfies the DCC.

Lemma 4.3. [CGS14, Lemma 2.3][MP04, Lemma 4.3][HMX14, Lemma 4.1] *Let $(X, \Delta \geq 0)$ be a log canonical pair such that the coefficients of Δ belong to a set $I \subseteq [0, 1]$. Let S be a normal irreducible component of $[\Delta]$ and $\Theta \geq 0$ be the \mathbb{Q} -divisor on S defined by adjunction:*

$$(K_X + \Delta)|_S = K_S + \Theta.$$

Then, the coefficients of Θ belong to $D(I)$.

Definition 4.4 (Divisorial part and Moduli part). [PS09, Section 7][CTX13, Section 6] Let $f : X \rightarrow Z$ be a surjective proper morphism between two normal varieties and $K_X + D \sim_{\mathbb{Q}} f^*L$, where D is a boundary divisor on X and L is a \mathbb{Q} -Cartier \mathbb{Q} -divisor on Z . Let (X, D) be LC near the generic fiber of f , i.e., $(f^{-1}U, D|_{f^{-1}U})$ is LC for some Zariski dense open subset $U \subseteq Z$. Then we define two divisors D_{div} and D_{mod} on Z in the following way:

$$D_{\text{div}} = \sum (1 - c_Q)Q, \text{ where } Q \subseteq Z \text{ are prime Weil divisors of } Z,$$

$$c_Q = \sup\{c \in \mathbb{R} : (X, D + cf^*Q) \text{ is LC over the generic point } \eta_Q \text{ of } Q\} \text{ and}$$

$$D_{\text{mod}} = L - K_Z - D_{\text{div}}, \text{ so that } K_X + D \sim_{\mathbb{Q}} f^*(K_Z + D_{\text{div}} + D_{\text{mod}}).$$

Remark 4.5. Observe that D_{div} is a fixed divisor on Z , called the *Divisorial part* and D_{mod} is a \mathbb{Q} -linear equivalence class on Z , called the *Moduli part*. For other properties of D_{div} and D_{mod} see [PS09, Section 7] and [Amb99, Section 3].

Let $\overline{\mathcal{M}}_{0,n}$ be the moduli space of n -pointed stable curves of genus 0, $f_{0,n} : \overline{\mathcal{U}}_{0,n} \rightarrow \overline{\mathcal{M}}_{0,n}$ the universal family, and $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$, the sections of $f_{0,n}$ which correspond to the marked points (see [Kee92] and [Knu83]). Let d_j ($j = 1, 2, \dots, n$) be rational numbers such that $0 < d_j \leq 1$ for all j , $\sum_j d_j = 2$ and $\mathcal{D} = \sum_j d_j \mathcal{P}_j$.

Lemma 4.6. (1) *There exists a smooth projective variety $\mathcal{U}_{0,n}^*$, a \mathbb{P}^1 -bundle $g_{0,n} : \mathcal{U}_{0,n}^* \rightarrow \overline{\mathcal{M}}_{0,n}$, and a sequence of blowups with smooth centers*

$$\overline{\mathcal{U}}_{0,n} = \mathcal{U}^{(1)} \xrightarrow{\sigma_2} \mathcal{U}^{(2)} \xrightarrow{\sigma_3} \dots \xrightarrow{\sigma_{n-2}} \mathcal{U}^{(n-2)} = \mathcal{U}_{0,n}^*.$$

- (2) *Let $\sigma : \overline{\mathcal{U}}_{0,n} \rightarrow \mathcal{U}_{0,n}^*$ be the induced morphism, and $\mathcal{D}^* = \sigma_*\mathcal{D}$. Then $K_{\overline{\mathcal{U}}_{0,n}} + \mathcal{D} - \sigma^*(K_{\mathcal{U}_{0,n}^*} + \mathcal{D}^*)$ is effective.*

(3) *There exists a semi-ample \mathbb{Q} -divisor \mathcal{L} on $\overline{\mathcal{M}}_{0,n}$ such that*

$$K_{\mathcal{U}_{0,n}^*} + \mathcal{D}^* \sim_{\mathbb{Q}} g_{0,n}^*(K_{\overline{\mathcal{M}}_{0,n}} + \mathcal{L}).$$

Proof. The proof in [Kaw97b, Theorem 2] works in positive characteristic without any change (see also [CTX13, 6.7], [PS09, 8.5] and [KMM94, Section 3]). \square

Lemma 4.7 (Stable Reduction Lemma). *Let B be a smooth curve and $f : X \rightarrow B$, a flat family of rational curves such that the general fiber is isomorphic to \mathbb{P}^1 , and a unique singular fiber X_0 over $0 \in B$. Also assume that $f|_{X^*} : (X^* = X \setminus X_0; \mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n) \rightarrow B^* = B - \{0\}$ is a flat family of n -pointed stable rational curves sitting in the following commutative diagram*

$$(4.1) \quad \begin{array}{ccc} X^* = B^* \times_{\overline{\mathcal{M}}_{0,n}} \overline{\mathcal{U}}_{0,n} & \longrightarrow & \overline{\mathcal{U}}_{0,n} \\ f|_{X^*} \downarrow & & \downarrow \\ B^* & \longrightarrow & \overline{\mathcal{M}}_{0,n} \end{array}$$

Then there exists a unique flat family $\hat{f} : \hat{X} \rightarrow B$ of n -pointed stable rational curves satisfying the following commutative diagram

$$(4.2) \quad \begin{array}{ccccc} X & \xleftarrow{\quad \quad \quad} & \hat{X} = B \times_{\overline{\mathcal{M}}_{0,n}} \overline{\mathcal{U}}_{0,n} & \longrightarrow & \overline{\mathcal{U}}_{0,n} \\ f \downarrow & & \hat{f} \downarrow & & \downarrow \\ B & \xleftarrow{\quad id_B \quad} & B & \longrightarrow & \overline{\mathcal{M}}_{0,n} \end{array}$$

where the broken horizontal map is a birational map such that $f^{-1}B^ \cong \hat{f}^{-1}B^*$.*

Proof. Since $\overline{\mathcal{M}}_{0,n}$ is a proper scheme, by the valuative criterion of properness any morphism $B^* \rightarrow \overline{\mathcal{M}}_{0,n}$ extends uniquely to a morphism $B \rightarrow \overline{\mathcal{M}}_{0,n}$. Now since $\overline{\mathcal{M}}_{0,n}$ has a universal family $\overline{\mathcal{U}}_{0,n}$, the existence of $\hat{f} : \hat{X} \rightarrow B$ follows by taking the fiber product. \square

Theorem 4.8 (Canonical Bundle Formula). *Let $f : X \rightarrow Z$ be a proper surjective morphism, where X is a normal surface and Z is a smooth curve over an algebraically closed field k of char $k > 0$. Assume that $Q = \sum_i Q_i$ is a divisor on Z such that f is smooth over $(Z - \text{Supp}(Q))$ with fibers isomorphic to \mathbb{P}^1 . Let $D = \sum_j d_j P_j$ be a \mathbb{Q} -divisor on X , where $d_j = 0$ is allowed, which satisfies the following conditions:*

- (1) $(X, D \geq 0)$ is KLT.
- (2) $D = D^h + D^v$, where $D^h = \sum_{f(D_j)=Z} d_j D_j$ and $D^v = \sum_{f(D_j) \neq Z} d_j D_j$. An irreducible component of D^h (resp. D^v) is called horizontal (resp. vertical) component.

- (3) $\text{char } k = p > \frac{2}{\delta}$, where δ is the minimum non-zero coefficient of D^h .
 (4) $K_X + D \sim_{\mathbb{Q}} f^*(K_Z + M)$ for some \mathbb{Q} -Cartier divisor M on Z .

Then there exist an effective \mathbb{Q} -divisor $D_{\text{div}} \geq 0$ and a semi-ample \mathbb{Q} -divisor $D_{\text{mod}} \geq 0$ on Z (as defined in 4.4) such that

$$K_X + D \sim_{\mathbb{Q}} f^*(K_Z + D_{\text{div}} + D_{\text{mod}}).$$

Proof. The sketch of the proof of this formula is given in [CTX13, 6.7]. We include a complete proof following the idea of the proof of [PS09, Theorem 8.1].

First we reduce the problem to the case where all components of D^h are sections. Let D_{i_0} be a horizontal component of D and $Z' \rightarrow D_{i_0}$ be the normalization of D_{i_0} . Then $\nu : Z' \rightarrow Z$ is a finite surjective morphism of smooth curves. Let X' be the normalization of the component of $X \times_Z Z'$ dominating Z' .

$$(4.3) \quad \begin{array}{ccc} X & \xleftarrow{\nu'} & X' \\ f \downarrow & & \downarrow f' \\ Z & \xleftarrow{\nu} & Z' \end{array}$$

Let $k = \deg(\nu : Z' \rightarrow Z)$ and l be a general fiber of f . Then

$$(4.4) \quad k = D_i \cdot l \leq \frac{1}{d_i}(D \cdot l) = \frac{1}{d_i}(-K_X \cdot l) = \frac{2}{d_i} \leq \frac{2}{\delta} < \text{char } k.$$

Therefore $\nu : Z' \rightarrow Z$ is a separable morphism.

Let D' be the log pullback of D under ν' , i.e.,

$$(4.5) \quad K_{X'} + D' = \nu'^*(K_X + D).$$

More precisely we have (by [Kol92, 20.2])

$$D' = \sum_{i,j} d'_{ij} D'_{ij}, \quad \nu'(D'_{ij}) = D_i, \quad d'_{ij} = 1 - (1 - d_i)e_{ij},$$

where e_{ij} 's are the ramification indices along the D'_{ij} 's.

By construction X dominates Z . Also, since ν is etale over a dense open subset of Z , say, $\nu^{-1}U \rightarrow U$, and etale morphisms are stable under base change, $(f' \circ \nu)^{-1}U \rightarrow f^{-1}U$ is etale. Thus the ramification locus Λ of ν' does not contain any horizontal divisor of f' , i.e., $f'(\Lambda) \neq Z'$. Therefore D' is a boundary near the generic fiber of f' , i.e., D'^h is effective. We observe that the coefficients of D'^h can be computed by intersecting with a general fiber of $f' : X' \rightarrow Z'$, hence they are equal to the coefficients of $D^h \subseteq X$. Thus the

condition $p > \frac{2}{\delta}$ remains true for D' on X' .

After finitely many such base changes let $g : \tilde{X} \rightarrow \tilde{Z}$ be a family such that all of the horizontal components of $D_{\tilde{X}}$ are sections of g , where $D_{\tilde{X}}$ is the log pullback of D , i.e., $K_{\tilde{X}} + D_{\tilde{X}} = \psi^*(K_X + D)$.

$$(4.6) \quad \begin{array}{ccc} X & \xleftarrow{\psi} & \tilde{X} \\ f \downarrow & & \downarrow g \\ Z & \xleftarrow{\psi_0} & \tilde{Z} \end{array}$$

By Lemma 4.7, we get a family of n -pointed stable rational curves $\tilde{X} = \tilde{Z} \times_{\overline{\mathcal{M}}_{0,n}} \overline{\mathcal{U}}_{0,n} \rightarrow \tilde{Z}$. Let X' be the common resolution of \tilde{X} and \hat{X} . Let $\hat{X} = \tilde{Z} \times_{\overline{\mathcal{M}}_{0,n}} \mathcal{U}_{0,n}^*$. By the universal property of fiber products there exists a morphism $\mu : X' \rightarrow \hat{X}$. Since X' , \tilde{X} and \hat{X} are all isomorphic \mathbb{P}^1 -bundles over a dense open subset $U \subseteq \tilde{Z}$, $\mu : X' \rightarrow \hat{X}$ is birational.

$$(4.7) \quad \begin{array}{ccccccc} & & X' & & & & \\ & \swarrow \pi & \downarrow \tilde{f} & \searrow \mu & \swarrow \hat{\phi} & & \\ X & \xleftarrow{\psi} & \tilde{X} & \dashrightarrow & \hat{X} & \dashrightarrow & \overline{\mathcal{U}}_{0,n} \xrightarrow{\sigma} \mathcal{U}_{0,n}^* \\ f \downarrow & & \downarrow g & & \downarrow \hat{f} & & \downarrow f_{0,n} \\ Z & \xleftarrow{\psi_0} & \tilde{Z} & \xrightarrow{\phi_0} & \overline{\mathcal{M}}_{0,n} & & \end{array}$$

(Note: The diagram also includes a curved arrow λ from X' to \tilde{X} , a curved arrow ϕ from X' to \hat{X} , and a diagonal arrow $g_{0,n}$ from $\mathcal{U}_{0,n}^*$ to $\overline{\mathcal{M}}_{0,n}$.)

Let D' and \hat{D} be \mathbb{Q} -divisors on X' and \hat{X} respectively, defined by

$$(4.8) \quad K_{X'} + D' = \pi^*(K_X + D).$$

and

$$K_{\hat{X}} + \hat{D} = \mu_*(K_{X'} + D').$$

Since $K_{X'} + D'$ is a pullback from the base \tilde{Z} (by (4.7)), by the Negativity lemma we get

$$(4.9) \quad K_{X'} + D' = \mu^*(K_{\hat{X}} + \hat{D}).$$

Since the definition of the *divisorial part* of the adjunction does not depend on the birational modification of the family (see [PS09, Remark 7.3(ii)] or [Amb99, Remark 3.1]), we will define it with respect to $\hat{f} : \hat{X} \rightarrow \tilde{Z}$. First we will show that the \mathbb{Q} -divisor \hat{D}_{mod} on \tilde{Z} is semi-ample.

Since $\hat{\phi}$ is finite and \mathcal{D}^* is horizontal it follows that $\hat{\phi}^*\mathcal{D}^*$ is horizontal too. Since \hat{D}^h is also horizontal one sees that

$$(4.10) \quad \hat{D}^h = \hat{\phi}^*\mathcal{D}^*.$$

From the construction of $\sigma : \overline{\mathcal{U}}_{0,n} \rightarrow \mathcal{U}_{0,n}^*$ we see that $(F, \mathcal{D}^*|_F)$ is log canonical for any fiber F of $g_{0,n} : \mathcal{U}_{0,n}^* \rightarrow \overline{\mathcal{M}}_{0,n}$. Since the fibers of $\hat{f} : \hat{X} \rightarrow \hat{Z}$ are isomorphic to the fibers of $g_{0,n}$, $(\hat{F}, \hat{D}^h|_{\hat{F}})$ is also log canonical, where \hat{F} is a fiber of \hat{f} . Finally, since \hat{X} is a surface, by inversion of adjunction $(\hat{X}, \hat{F} + \hat{D}^h)$ is log canonical near \hat{F} . Thus, since the fibers of \hat{f} are reduced, the lct of $(\hat{X}, \hat{D}; \hat{F})$ over the generic point of \hat{F} is $(1 - \text{coeff.}_{\hat{F}} \hat{D})$. Hence we get

$$(4.11) \quad \hat{D}^v = \hat{f}^* \hat{D}_{\text{div}}.$$

By definition of \hat{D}_{mod} we have

$$(4.12) \quad K_{\hat{X}} + \hat{D}^h \sim_{\mathbb{Q}} \hat{f}^*(K_{\hat{Z}} + \hat{D}_{\text{mod}}).$$

Then we have

$$(4.13) \quad K_{\hat{X}} + \hat{D}^h - \hat{f}^*(K_{\hat{Z}} + \phi_0^*\mathcal{L}) = K_{\hat{X}/\hat{Z}} + \hat{D}^h - \hat{\phi}^*K_{\mathcal{U}_{0,n}^*/\overline{\mathcal{M}}_{0,n}} - \hat{\phi}^*\mathcal{D}^* \sim_{\mathbb{Q}} 0,$$

where the first equality follows from (4.12) and Lemma 4.6, and the second relation from (4.10) and [Liu02, Chapter 6, Theorem 4.9 (b) and Example 3.18].

Since \hat{f} has connected fibers, by (4.12) and (4.13) and the projection formula for locally free sheaves, we get

$$\hat{D}_{\text{mod}} \sim_{\mathbb{Q}} \phi_0^*\mathcal{L}$$

i.e., \hat{D}_{mod} is semi-ample.

Now, since $\psi_0 : \tilde{Z} \rightarrow Z$ is a composition of finite morphisms of degree strictly less than $\text{char } k$, by [Kol13, Corollary 2.43] and [Amb99, Theorem 3.2] (also see [CTX13, 6.6]) we get

$$(4.14) \quad K_{\tilde{Z}} + \hat{D}_{\text{div}} \sim_{\mathbb{Q}} \psi_0^*(K_Z + D_{\text{div}}).$$

Therefore

$$(4.15) \quad \psi_0^*D_{\text{mod}} \sim_{\mathbb{Q}} \hat{D}_{\text{mod}}$$

Since Z and \tilde{Z} are both smooth curves, D_{mod} is semi-ample. □

Theorem 4.9. *Let $(X, D \geq 0)$ be a \mathbb{Q} -factorial 3-fold log canonical pair such that the coefficients of D are contained in a DCC set $I \subseteq [0, 1]$. Let W be a minimal log canonical center of (X, D) , and assume that the codimension of*

W is 2. Also assume that X has KLT singularities and $\text{char } k > \max\{5, \frac{2}{\delta}\}$, where δ is the non-zero minimum of the set $D(I)$ (defined in 4.1). Then the following hold:

- (1) W is normal.
- (2) There exists effective \mathbb{Q} -divisors D_W and M_W on W such that $(K_X + D)|_W \sim_{\mathbb{Q}} K_W + D_W + M_W$. Moreover, if $D = D' + D''$ with D' (resp. D'') the sum of all irreducible components which contain (resp. do not contain) W , then M_W is determined only by the pair (X, D') .
- (3) There exists an effective \mathbb{Q} -divisor M'_W such that $M'_W \sim_{\mathbb{Q}} M_W$ and the pair $(W, D_W + M'_W)$ is KLT.

Proof. Normality of W follows from Theorem 3.6.

Since X is \mathbb{Q} -factorial, $(K_X + D)|_W = (K_X + D' + D'')|_W = (K_X + D')|_W + D''|_W$. Thus we may assume that all the components of D contain W . Since W is a minimal log canonical center of (X, D) and $\text{codim}_X W = 2$, it does not intersect any other LC center of codimension ≥ 2 , by Lemma 2.1. Thus by shrinking X (removing closed subsets of codimension ≥ 2 which do not intersect W) if necessary we may assume that W is the unique log canonical center of codimension ≥ 2 of (X, D) .

Let $f : (X', D') \rightarrow (X, D)$ be a \mathbb{Q} -factorial DLT model over (X, D) such that

$$(4.16) \quad K_{X'} + D' = f^*(K_X + D).$$

Such f exists by [Bir13, 7.7].

Note that, since X is \mathbb{Q} -factorial, the exceptional locus of f supports an effective f -anti-ample divisor. In particular all positive dimensional fibers of f are contained in the support of $[D']$.

Let E be an exceptional divisor dominating W . Then E is normal by Proposition 2.2. By adjunction we have

$$(4.17) \quad K_E + D'_E = (K_{X'} + D')|_E = f|_E^*((K_X + D)|_W)$$

and (E, D'_E) is DLT, by Proposition 2.2 and the coefficients of D'_E are in the set $D(I)$ by Lemma 4.3.

By Theorem 4.8, there exist \mathbb{Q} -divisors $D_W = D_{\text{div}} \geq 0$ and $M_W = D_{\text{mod}} \geq 0$ on W such that

$$(4.18) \quad K_E + D'_E \sim_{\mathbb{Q}} f|_E^*(K_W + D_W + M_W).$$

Since $f|_E : E \rightarrow W$ has connected fibers, from (4.17), (4.18) and the projection formula for locally free sheaves, we get

$$(4.19) \quad (K_X + D)|_W \sim_{\mathbb{Q}} K_W + D_W + M_W.$$

Lemma 4.10 given below shows that D_W is independent of the choice of the exceptional divisor E dominating W .

From the definition of D_W we see that $D_W \geq 0$, since $D'_E \geq 0$. Also, since D_W is independent of the birational modifications (by [PS09, Remark 7.3(ii)]) and W is a minimal LC center, by taking a log resolution of (X', D') and working on the strict transform of E , we see that the coefficients of D_W are strictly less than 1. Thus $\lfloor D_W \rfloor = 0$.

Since M_W is semi-ample and W is a smooth curve, either $M_W = 0$ or M_W is ample. In the later case by Bertini's theorem there exists an effective \mathbb{Q} -divisor $M'_W \sim_{\mathbb{Q}} M_W$ such that $\lfloor M'_W \rfloor = 0$ and $\text{Supp}(M'_W) \cap \text{Supp}(D_W) = \emptyset$. Hence $(W, D_W + M'_W)$ is KLT. □

Lemma 4.10. *With the same hypothesis as in Theorem 4.9, the divisor $D_W = D_{\text{div}}$ on W is independent of the choice of the exceptional divisors dominating W .*

Proof. Let E_1 and E_2 be two exceptional divisors of f dominating W such that

$$(4.20) \quad K_{X'} + E_1 + E_2 + \Delta' = f^*(K_X + D),$$

where $f : X' \rightarrow X$ is the DLT model as above and $D' = E_1 + E_2 + \Delta'$.

Notice that if η_W is the generic point of W , then $f^*\eta_W \cap \text{NKLT}(X', E_1 + E_2 + \Delta)$ is connected (By localizing at η_W , this follows from a surface computation involving relative Kawamata-Viehweg vanishing theorem). Therefore we may assume that $E_1 \cap E_2 \neq \emptyset$.

By adjunction on E_1 we get

$$(4.21) \quad K_{E_1} + C + \Delta'_{E_1} = f|_{E_1}^*((K_X + D)|_W),$$

where C is an irreducible component of $E_1 \cap E_2$ dominating W .

Adjunction on C gives

$$(4.22) \quad K_C + \Delta'_C = f|_C^*((K_X + D)|_W).$$

Let Q be a point on W , and $t = \text{lct}(E_1, C + \Delta'_{E_1}; f|_{E_1}^*Q)$ and $s = \text{lct}(C, \Delta'_C; f|_C^*Q)$. Since C is an irreducible component of $E_1 \cap E_2$ dominating W , it is enough to

show that $t = s$. By adjunction, $t \leq s$. So by contradiction assume that $t < s$.

Since $(E_1, C + \Delta'_{E_1})$ is DLT by Proposition 2.2, $(E_1, C + \Delta'_{E_1} + t'f|_{E_1}^*Q)$ is LC outside of $f|_{E_1}^{-1}Q$ for any $t' > t$. Thus all NLC centers of $(E_1, C + \Delta'_{E_1} + t'f|_{E_1}^*Q)$ appear along $f|_{E_1}^{-1}Q$.

The general fiber of $f|_{E_1} : E_1 \rightarrow W$ is isomorphic to \mathbb{P}^1 . Thus $\text{degree}((C + \Delta'_{E_1})|_{\mathbb{P}^1}) = 2$ by (4.21). There are two cases depending on whether C intersects the general fiber with degree 1 or 2.

Case I: C intersects the general fiber with degree 1. Then there exists a horizontal component C' of Δ'_{E_1} . Let H be an ample divisor on E_1 , and F_η , the generic fiber of $f|_{E_1} : E_1 \rightarrow W$. Choose $\lambda > 0$ such that

$$(H - \lambda C') \cdot F_\eta = 0.$$

Then $(H - \lambda C')|_{F_\eta} \sim_{\mathbb{Q}} 0$. Thus by [Cor07, 8.3.4], $H \sim_{\mathbb{Q}} \lambda C' - \sum \lambda_i F_i$, where the F_i 's are irreducible components of some fibers of $f|_{E_1}$. By adding the pullback of some appropriate divisors from the base to $\lambda C' - \sum \lambda_i F_i$, we may assume that $\lambda_i > 0$ for all i and $\lambda C' - \sum \lambda_i F_i$ is $f|_{E_1}$ -ample.

Assume that there exists a point $P \in f|_{E_1}^{-1}Q$ but $P \notin C$ such that $(E_1, C + \Delta'_{E_1} + (t + \epsilon)f|_{E_1}^*Q)$ is not LC at P , where $0 < \epsilon \ll 1$ such that $t + \epsilon < s$. Then by choosing $0 < \lambda, \lambda_i \ll 1$ we can assume that $(C + \Delta'_{E_1} - \lambda C' + \sum \lambda_i F_i) \geq 0$, $(E_1, C + \Delta'_{E_1} - \lambda C' + \sum \lambda_i F_i + (t + \epsilon)f|_{E_1}^*Q)$ is still not LC at P , and

$$(4.23) \quad -(K_{E_1} + C + \Delta'_{E_1} - \lambda C' + \sum \lambda_i F_i) = -f|_{E_1}^*((K_X + D)|_W) + (\lambda C' - \sum \lambda_i F_i)$$

is $f|_{E_1}$ -ample.

Then by [Bir13, 8.3], $\text{NKLT}(E_1, C + \Delta'_{E_1} - \lambda C' + \sum \lambda_i F_i + (t + \epsilon)f|_{E_1}^*Q) \cap f|_{E_1}^{-1}Q$ is connected. Let $R \in C \cap f|_{E_1}^{-1}Q$. Then there exists a chain of curves G_i 's connecting R and P , and contained in $\text{NKLT}(E_1, C + \Delta'_{E_1} - \lambda C' + \sum \lambda_i F_i + (t + \epsilon)f|_{E_1}^*Q) \cap f|_{E_1}^{-1}Q$.

Now $\text{NKLT}(E_1, C + \Delta'_{E_1} - \lambda C' + \sum \lambda_i F_i + (t + \epsilon)f|_{E_1}^*Q) \subseteq \text{NKLT}(E_1, C + \Delta'_{E_1} + \sum \lambda_i F_i + (t + \epsilon)f|_{E_1}^*Q)$. Since we are only concentrating on the NKLT centers along $f|_{E_1}^{-1}Q$, we may assume that F_i 's are all contained in $f|_{E_1}^{-1}Q$. Then by choosing $0 < \lambda_i \ll 1$ for all i , such that $t + \epsilon' = t + \epsilon + \max\{\lambda_i\} < s$, we see that $\text{NKLT}(E_1, C + \Delta'_{E_1} + \sum \lambda_i F_i + (t + \epsilon)f|_{E_1}^*Q) \subseteq \text{NKLT}(E_1, C + \Delta'_{E_1} + (t + \epsilon')f|_{E_1}^*Q)$. Thus the curves G_i 's are contained in the $\text{NKLT}(E_1, C + \Delta'_{E_1} + (t + \epsilon')f|_{E_1}^*Q)$. Hence G_i 's are contained in $\text{NLC}(E_1, C + \Delta'_{E_1} + sf|_{E_1}^*Q)$. This implies that $(E_1, C + \Delta'_{E_1} + sf|_{E_1}^*Q)$ is not LC at $R \in C$. Then by inversion

of adjunction we get a contradiction to the fact that $(C, \Delta'_C + sf|_C^*Q)$ is LC.

Case II: C intersects the general fiber with degree 2. In this case $E_1 \cap E_2 = C$, and Δ'_{E_1} and Δ'_{E_2} do not have any horizontal component with respect to $f|_{E_1}$ and $f|_{E_2}$, respectively, where Δ'_{E_1} and Δ'_{E_2} are defined by the adjunction

$$K_{E_1} + \Delta'_{E_1} = f|_{E_1}^*((K_X + \Delta)|_W) \quad \text{and} \quad K_{E_2} + \Delta'_{E_2} = f|_{E_2}^*((K_X + \Delta)|_W).$$

Since $D \neq 0$ and every component of D contains W , this implies that Δ' does not contain any component of $f_*^{-1}D$ (otherwise Δ'_{E_i} will have a non zero horizontal component with respect to $f|_{E_i}$). Therefore one of the E_i 's must be a component of $f_*^{-1}D$, say $E_2 = f_*^{-1}D_i$, where D_i is an irreducible component of D . Thus in this case the exceptional divisors of f do not intersect each other. Since X is \mathbb{Q} -factorial, the exceptional locus $\text{Ex}(f)$ of $f : X' \rightarrow X$ supports an effective f -anti-ample divisor and hence $\text{Ex}(f) \cap f^{-1}(w)$ is connected for all $w \in W$. Thus f has a unique exceptional divisor in this case and we are done.

□

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